Note On Property (A_1)

Hari Bercovici*
Department of Mathematics
Indiana University
Bloomington, Indiana 47401

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ABSTRACT

It is shown that there is a subspace N of $\mathbf{M}_3$, the space of 3×3 matrices, such that the bilinear map $x\otimes y$ from $\mathbb{C}^3\times\mathbb{C}^3$ to $\mathbf{M}_3$ mod N is a (continuous) surjection, but not open at the origin. With $\mathbf{M}_7$ instead of $\mathbf{M}_3$, N can be chosen such that the annihilator of N, with respect to the trace pairing, is a commutative algebra.

Let $L(H)$ denote the algebra of bounded linear operators acting on the complex Hilbert space H, and let $M\subset L(H)$ be an arbitrary linear subspace. We will denote by M_* the collection of all weak* continuous functionals on M; note that M_* is the usual dual of M if H, and hence M, is finite-dimensional. Now, $L(H)_*$ can be identified with the space $C_1(H)$ of all trace-class operators on H via the pairing $\langle T, A\rangle=\operatorname{tr}(AT)$, $A\in L(H)$, $T\in C_1(H)$. Therefore M_* can be identified with the quotient space $Q_M=C_1(H)/M_\perp$, where

$$M_\perp=\{T\in C_1(H):\langle T, A\rangle=0\text{ for all }A\in M\}.$$

For arbitrary vectors $x, y\in H$ one defines a functional $[x\otimes y]\in M_*$ via

$$[x\otimes y](A)=(Ax|y),\qquad A\in M,$$

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where $(\cdot|\cdot)$ denotes the scalar product in H. The notation $[x\otimes y]$ is justified because, under the identification $M_* \simeq Q_M$, $[x\otimes y]$ corresponds with the class of the rank-one operator $x\otimes y \in L(H)$, where $(x\otimes y)(h) = (h|y)x$, $h\in H$.

DEFINITION 1. The space M is said to have property $(\mathbb{A}_1)$ if every functional f in M_* can be written as $f=[x\otimes y]$ for some $x, y\in H$. Let r be a positive real number. Then M is said to have property $(\mathbb{A}_1(r))$ if for every $f\in M_*$ and every $s>r$ there exist $x, y\in H$ satisfying $f=[x\otimes y]$ and $\|x\|\,\|y\|\leqslant s\|f\|$. Finally, M is said to have property $(\mathbb{A}_{1/2}(r))$ if for every $f\in M_*$ and $s>r$ there exist $x, y, x', y'\in H$ satisfying $f=[x\otimes y]+[x'\otimes y']$ and $(\|x\|^2+\|x'\|^2)(\|y\|^2+\|y'\|^2)\leqslant s^2\|f\|^2$.

These definitions, and a discussion of their relevance in the study of operators and operator algebras, are given in [1]. Properties $(\mathbb{A}_1)$ and $(\mathbb{A}_1(r))$ appear under the names (D_σ) and $(D_\sigma(r))$, respectively, in [5].

It was proved by Chevreau and Esterle [2] and, independently, by Dixon [4] that property $(\mathbb{A}_1)$ always implies property $(\mathbb{A}_{1/2}(r))$ for some $r>0$. In this note we show that property $(\mathbb{A}_1)$ does not imply property $(\mathbb{A}_1(r))$ for any $r>0$, even for algebras $M\subset L(H)$. Observe that property $(\mathbb{A}_1)$ simply means that the sesquilinear map $(x,y)\to[x\otimes y]$ is onto M_*, while $(\mathbb{A}_1(r))$ for some $r>0$ means that this map is open at the origin. It has been known for some time that continuous surjective bilinear maps between Banach spaces need not be open at the origin (cf. [3] and [6]). One might however have expected that maps of the form $(x,y)\to[x\otimes y]$ possess some special properties which would make $(\mathbb{A}_1)$ imply $(\mathbb{A}_1(r))$ for some $r>0$. We begin by showing that these special bilinear maps are in fact quite general.

THEOREM 2. *Let H be a Hilbert space, X a Banach space, and $b\colon H\times H\to X$ a continuous sesquilinear map such that the set $\{b(x,y)\colon x,y\in H\}$ spans X linearly [i.e., every vector in X is a finite linear combination of elements of the form $b(x,y)$]. Then there exists a weak* closed linear subspace $M\subset L(H)$ and a continuous invertible linear map $c\colon M_*\to X$ such that $b(x,y)=c([x\otimes y])$, $x,y\in H$.*

Proof. We recall that $C_1(H)$ can be identified with the projective tensor product of H with its conjugate version $\overline{H}$. It follows that we can find a continuous linear map $p\colon C_1(H)\to X$ such that $p(x\otimes y)=b(x,y)$ for $x,y\in H$. Note that the range of p contains the range of b, and hence p is onto. It follows that the linear map $c\colon C_1(H)/\ker(p)\to X$ defined by

$c(A+\ker(p)) = p(A)$, $A \in C_1(H)$, is invertible. It suffices now to define

$$M = \{T \in L(H) : \langle T, A\rangle = 0 \text{ for every } A \in \ker(p)\}.$$

Indeed, one sees easily from the Hahn-Banach theorem that $M_\perp = \ker(p)$ and hence $M_* \simeq C_1(H)/\ker(p)$. The equality $b(x, y) = c([x \otimes y])$ is immediate, and the theorem follows. ■

Corollary 3. *There exists a four-dimensional subspace $M \subset L(\mathbb{C}^3)$ which has property $(\mathbb{A}_1)$ but fails to have property $(\mathbb{A}_1(r))$ for any $r > 0$.*

Proof. Horowitz showed in [6] that the map $b: \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^4$ given by $b(x, y) = (x_1\bar{y}_1, x_1\bar{y}_2, x_1\bar{y}_3 + x_3\bar{y}_1 + x_2\bar{y}_2, x_3\bar{y}_2 + x_2\bar{y}_1)$, $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, is onto, but not open at the origin. The subspace M associated with b in the preceding theorem is the required space. It is easy to see that M has dimension 4. Indeed, $\dim(M) = 9 - \dim(\ker(p)) = \dim(\operatorname{ran}(p)) = \dim(\mathbb{C}^4) = 4$. The corollary follows. ■

Corollary 4. *There exists a commutative five-dimensional algebra $A \subset L(\mathbb{C}^7)$ which has property $(\mathbb{A}_1)$ but fails to have property $(\mathbb{A}_1(r))$ for any $r > 0$.*

Proof. We identify $\mathbb{C}^7$ with $\mathbb{C}^3 \oplus \mathbb{C}^3 \oplus \mathbb{C}$, and operators on $\mathbb{C}^7$ with 3 by 3 block matrices. Let M be as in Corollary 3, and set

$$A = \left\{ \begin{bmatrix} wI & T & 0 \\ 0 & wI & 0 \\ 0 & 0 & w \end{bmatrix} : w \in \mathbb{C},\ T \in M \right\},$$

where I denotes the identity operator on $\mathbb{C}^3$. It is obvious that A is a commutative algebra of dimension 5. Moreover, if $x \oplus y \oplus z$, $x' \oplus y' \oplus z' \in \mathbb{C}^7$, $T \in M$, and $w \in \mathbb{C}$, we have

$$\left(\begin{bmatrix} wI & T & 0 \\ 0 & wI & 0 \\ 0 & 0 & w \end{bmatrix} (x \oplus y \oplus z) \middle| x' \oplus y' \oplus z' \right)$$

$$= w((x|x') + (y|y') + z\bar{z}') + [x' \otimes y](T).$$

It is easy to see now that A has property $(\mathbb{A}_1)$. Indeed, an arbitrary functional g on A must have the form

$$g\left(\begin{bmatrix} wI & T & 0 \\ 0 & wI & 0 \\ 0 & 0 & w \end{bmatrix}\right) = aw + f(T)$$

for some $a \in \mathbb{C}$ and $f \in M_*$. Choose then $x = 0$, $y' = 0$, $z = a$, $z' = 1$, and x', y such that $[x' \otimes y] = f$. The calculation above shows that $[(x \oplus y \oplus z) \otimes (x' \oplus y' \oplus z')] = g$. Note however that if $a = 0$, and $x \oplus y \oplus z$ and $x' \oplus y' \oplus z'$ are such that $[(x \oplus y \oplus z) \otimes (x' \oplus y' \oplus z')] = g$ and $\|x \oplus y \oplus z\| \, \|x' \oplus y' \oplus z'\| \leqslant s\|g\|$, then necessarily $[x' \otimes y] = f$ and $\|x'\| \, \|y\| \leqslant s\|f\|$. Since M does not have property $(\mathbb{A}_1(r))$ for any $r > 0$, it follows that A also fails to have any of these properties. The corollary is proved. ■

REFERENCES

1 H. Bercovici, C. Foiaş, and C. Pearcy, *Dual Algebras with Applications to Invariant Subspaces and Dilation Theory*, CBMS Conf. Ser. in Math., No. 56, Amer. Math. Soc., Providence, 1985.
2 B. Chevreau, and C. Esterle, private communication.
3 P. Cohen, A counterexample to the closed graph theorem for bilinear maps, *J. Funct. Anal.* 16:235–239 (1974).
4 P. Dixon, private communication.
5 D. Hadwin and E. Nordgren, Subalgebras of reflexive algebras, *J. Operator Theory* 7:3–23 (1982).
6 C. Horowitz, An elementary counterexample to the open mapping principle for bilinear maps, *Proc. Amer. Math. Soc.* 53:293–294 (1975).

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